

Input-output theory of Coulomb Blockade

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We present an input-output formalism describing a tunnel junction strongly coupled to its electromagnetic environment. We exploit it in order to investigate the dynamics of the radiation being emitted and scattered by the junction. We find that the non-linearity imprinted in the electronic transport by a properly designed environment generates strongly squeezed radiation. Our results show that the interaction between a quantum conductor and electromagnetic fields can be exploited as a resource to design simple sources of non-classical radiation.

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Circuit quantum electrodynamics (cQED) describes at a quantum level the interaction between electromagnetic fields and artificial atoms implemented by quantum conductors such as Josephson junctions [1] or quantum dots [2–4]. This new architecture has triggered a number of pioneering experiments [5–8]. However quantum conductors can also be continuously driven out-of-equilibrium by dc biases, giving rise to situations having no evident counterpart in atomic physics. Recent predictions and experiments relevant to these situations already point to interesting quantum electrodynamics effects. To cite a few, dc driven quantum conductors can be used as stochastic amplifying media giving rise to lasing (or masing) transitions in the field stored in RF cavities [9–11], as sources of sub-poissonian [12–16] and squeezed radiation [17–26]. Conversely, non-classical features of an incoming field may be revealed in the $I(V)$ curves of the conductor [27].

The common underlying mechanism for these effects is the probabilistic transfer of discrete charge carriers through the quantum conductors. The resulting current fluctuations excite the surrounding electromagnetic environment. This coupling not only results in photon emission, but also modifies the transport properties of the conductor itself [28–37], an effect known as Dynamical Coulomb Blockade (DCB) [38]. By increasing the impedance of the electromagnetic environment to values comparable to the resistance quantum $R_K = h/e^2 \simeq 25.9 \text{ k}\Omega$, the resulting strong coupling suppresses the transport at low bias for a normal conductor [28–30, 35–37].

So far most DCB studies focused on the conductor's transport properties at low frequencies. Recent progress in microwave techniques opens new perspectives for investigating the quantum properties of the emitted radiation [11, 23, 34, 39]. Such radiative properties lie out of the scope of the standard DCB approach, yet one could expect the emission of non-classical radiation in a strong

coupling regime. In the case of a Josephson junction, it has indeed been predicted [40, 41] that the emitted photons are strongly antibunched. In this letter, we consider the case of a normal tunnel junction arbitrarily coupled to radiation. We develop a Hamiltonian approach which considers not only the electric transport through the conductor but also the associated radiative dynamics via an input-output description [42]. Exploiting this approach, we predict that a normal tunnel junction in the strong DCB regime can efficiently squeeze radiation. Squeezing in this scheme is induced by the dissipative bath provided by the tunnel junction [43, 44].

Standard DCB - We first consider the circuit shown in Figure 1 (a), and treat it in the standard DCB formulation [38]. It contains a tunnel element shunted by an LC circuit of resonant frequency $\omega_0 = 1/\sqrt{LC}$ and characteristic impedance $Z_{LC} = \sqrt{L/C}$. The electrodynamic coupling gives rise to inelastic tunneling events, modifying the charge transfer dynamics of the junction. This physics is described by the Hamiltonian $H_0 = H_{qp} + H_T + H_{LC}$, with $H_{qp} = \sum_l \epsilon_l c_l^\dagger c_l + \sum_r \epsilon_r c_r^\dagger c_r$ describing the left and right electrodes of the junction, $H_{LC} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$ the energy stored in the LC circuit, and $H_T = T + T^\dagger$ with $T = \sum_{l,r} \tau_{l,r} c_l^\dagger c_r e^{ie\Phi/\hbar}$ the tunnel coupling which simultaneously transfers quasiparticles from the right to the left electrode with amplitude $\tau_{l,r}$, while displacing the influence charge of the capacitance by the electron charge $e^{ie\Phi/\hbar} Q e^{-ie\Phi/\hbar} = Q - e$. H_T is the minimal coupling of the junction to its circuit, neglecting the intrinsic electrodynamics of the electrodes [15, 45, 46] beyond the mean-field approximation encompassed in the shunting capacitance C .

From this Hamiltonian we obtain the quasiparticle current $I_{qp} = d(e \sum_l (c_l^\dagger c_l))/dt = \frac{ie}{\hbar} (T^\dagger - T)$, the displacement current $I_D = dQ/dt = \frac{ie}{\hbar} (T^\dagger - T) - \Phi/L$, and the inductive current $I_L = \Phi/L$ which correctly compensate at the circuit node $I_{qp} = I_D + I_L$ as required by gauge invariance. We now consider the radiative proper-

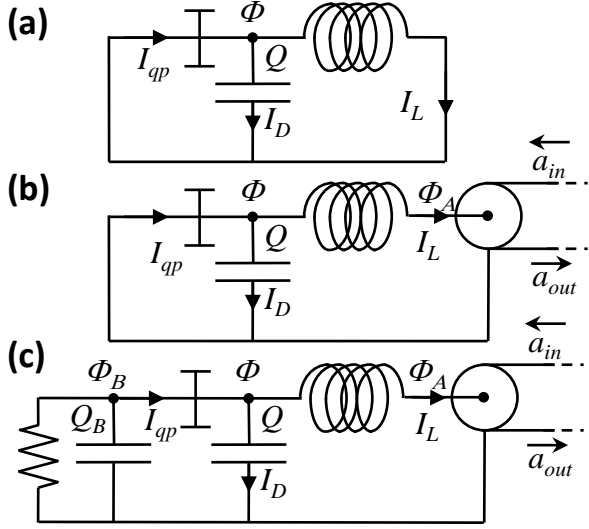


FIG. 1: Schemes of the considered circuits: (a) A tunnel junction with quasiparticle current I_{qp} is embedded in an LC circuit described by conjugated fields at the depicted node: the inductive magnetic flux Φ , and the capacitive influence charge Q . (b) A transmission line in series with the inductor damps the circuit by radiating outgoing modes a_{out} , which can be detected with a matched detection chain (not shown). (c) A high resistance ($R > R_K$) RC circuit is connected to the junction. The resulting quantum flux fluctuations $\Phi_B(t)$ are responsible for strong dynamical Coulomb blockade modifying the quasiparticle current I_{qp} , which can be efficiently collected in a matched detection band of the transmission line.

ties in the presence of both dc and ac bias, by evaluating the power emitted in the LC circuit $P_{LC} = dH_{LC}/dt = (I_{qp}Q + Q\dot{I}_{qp})/2C$ and computing its expectation value to lowest order in the tunnel coupling H_T . To do so, we take the uncoupled boundary condition for the density matrix $\rho = \rho_{qp} \otimes \rho_{LC}$. The electrodes are initially at thermal equilibrium $\rho_{qp} = e^{-\beta H_{qp}}/Z_{qp}$. The LC circuit is set in a displaced thermal state $\rho_{LC} = D[\gamma]e^{-\beta H_{LC}}/Z_{LC}D^\dagger[\gamma]$ [47, 48] where the displacement vector $\gamma = \frac{ieV_{ac}}{2r\hbar\omega_0}$, with $r = \sqrt{\pi e^2 Z_{LC}/\hbar}$, gives the deterministic voltage: $\langle V(t) \rangle = \text{Tr}(\rho_{LC} d\Phi/dt) = V_{ac} \cos(\omega_0 t) + V_{dc}$. The average power reads [49]:

$$\langle P_{LC}(t) \rangle = \frac{(1 + n_B(\hbar\omega_0))S_{I_{qp}}(\omega_0, t) - n_B(\hbar\omega_0)S_{I_{qp}}(-\omega_0, t)}{2C} - \langle I_{qp}(t) \rangle V_{ac} \cos(\omega_0 t), \quad (1)$$

where $n_B(\hbar\omega_0)$ is the bosonic thermal population of the LC mode, $S_{I_{qp}}(\omega, t) = \int d\tau e^{-i\omega\tau} \langle I_{qp}(t+\tau)I_{qp}(t) \rangle$ is the power spectral density of quasiparticle current fluctuations [48] (emission noise being here at positive frequency), and $\langle I_{qp}(t) \rangle$ is the average quasiparticle current. In this expression both $S_{I_{qp}}(\omega_0, t)$ and $\langle I_{qp}(t) \rangle$ have an explicit time-dependence due to the ac bias. The first term in Eq. (1) describes the power being emitted/absorbed by the junction via its emission/absorption

current fluctuations. It has the same structure as that derived neglecting the quantum and deterministic voltage fluctuations [50, 51], the important difference being that the tunneling dynamics encompassed in $S_{I_{qp}}(\omega_0, t)$ take into account DCB and photon-assisted tunneling effects.

The second term describes the Joule power dissipated in the junction via its mean current response in phase with the ac excitation. Indeed, computing the power injected in the electrodes $P_{qp} = dH_{qp}/dt$ [49] confirms that its average value is equal to the electrical power delivered by the dc source, minus that carried away by the LC circuit which acts both as a power source and sink: $\langle P_{qp}(t) \rangle = \langle I_{qp}(t) \rangle V_{dc} - \langle P_{LC}(t) \rangle$. This perturbative approach, valid in the high tunneling resistance limit, considers flux (voltage) fluctuations arising only from the external circuit dynamics. Moreover, it implicitly assumes the presence of additional mechanisms not specified in the Hamiltonian H_0 , restoring the initial state of the full system in between every tunneling event.

Circuit model for dissipation - We now include such a dissipative channel in the model, see Fig. 1(b), by adding a semi-infinite transmission line [42, 52] characterized by the impedance Z_0 . It has the advantage of providing (i) a precise mechanism for the damping of the LC circuit and (ii) an efficient way to compute the properties of the radiation emitted by the junction into a linear detection circuit using an input-output approach. Our analysis thus extends previous works [26, 53] by considering both quasiparticles and strong (DCB) backaction. We find in particular that the usual DCB formulation of Fig. 1(a) can be quantitatively understood as a separation of time scales: the LC resonator leaks photons in the transmission line much faster than it exchanges photons with the tunnel junction, corresponding to an impedance mismatch to the readout circuit $R_T \gg Z_{LC}^2/Z_0$, R_T denoting the tunnel junction resistance. The dynamics of the transmission line modifies the energy stored in the inductance $\frac{(\Phi - \Phi_A)^2}{2L}$ and is described by the Hamiltonian [42]:

$$H_{line} = \int_0^{+\infty} dx \left[\frac{1}{2\ell} \left(\frac{\partial \Phi_A(x)}{\partial x} \right)^2 + \frac{q_A(x)^2}{2c} \right], \quad (2)$$

with $\Phi_A \equiv \Phi_A(0)$, where we introduced the conjugated variables Φ_A and q_A describing the flux and charge density in the line. ℓ and c stand respectively for its lineic inductance and capacitance and x is the position along the line. The bosonic operators describing the input $a_{in,\omega}$ and output $a_{out,\omega}$ fields enter the mode decomposition of Φ_A ($Z_0 = \sqrt{\ell/c}$),

$$\Phi_A(x) = \sqrt{\frac{\hbar Z_0}{8\pi^2}} \int_0^{+\infty} \frac{d\omega}{\sqrt{\omega}} [a_{in,\omega} e^{-ikx} + a_{out,\omega} e^{ikx} + \text{h.c.}]. \quad (3)$$

The input-output theory is obtained by considering the time-evolution equations in the interaction representa-

tion. The Heisenberg equations coupling the fields of the line, the flux of the LC circuit and the quasiparticle current give

$$C\partial_t^2\Phi = \frac{\Phi_A - \Phi}{L} + \hat{I}_{qp}^H, \quad \frac{1}{\ell}\frac{\partial\Phi_A}{\partial x} = \frac{\Phi_A - \Phi}{L}, \quad (4)$$

where the Heisenberg current operator expands as

$$\hat{I}_{qp}^H(t) = \hat{I}_{qp}(t) + \frac{i}{\hbar} \int_{-\infty}^t dt' [H_T(t'), \hat{I}_{qp}(t)]. \quad (5)$$

Solving for equations (4) in frequency domain, one arrives at

$$a_{\text{out},\omega} = \frac{\Delta^*(\omega)}{\Delta(\omega)} a_{\text{in},\omega} - i\omega_0^2 \sqrt{\frac{2Z_0}{\hbar\omega}} \frac{\hat{I}_{qp}^H(\omega)}{\Delta(\omega)}, \quad (6)$$

with $\Delta(\omega) = \omega^2 - \omega_0^2 + i\omega\kappa$. $\kappa = \frac{Z_0}{L}$ is the LC resonator damping rate due to the transmission line. The first term corresponds to the input field reflected with a phase shift (time delay) by the resonator. The second term is the field emitted by the tunnel junction itself and carrying current noise fluctuations. As such Eq. (6) does not fully solve the circuit dynamics since the output field still enters the current \hat{I}_{qp}^H in the flux Φ dressing the tunneling operator T , calling for a self-consistent solution.

However, writing the rescaled flux $\tilde{\Phi}(\omega) = \sqrt{2\omega/(\hbar Z_0\omega_0^4)}\Phi(\omega)$ as

$$\tilde{\Phi}(\omega) = -(2a_{\text{in},\omega} + L\sqrt{2\omega/(\hbar Z_0)}\hat{I}_{qp}^H(\omega))/\Delta(\omega) \quad (7)$$

for $\omega \gg \kappa$, we find that the flux fluctuations arising from the second term, which calls for the self-consistency, are negligible in the case of strong impedance mismatch $S_{I_{qp}}(\omega_0)L^2\omega_0/(\hbar Z_0) \sim Z_{LC}^2/(R_T Z_0) \ll 1$. This can also be formulated in the time domain by inspecting the dynamics of the LC resonator: photons will leak much faster to the transmission line rather than to the tunnel junction [25, 27], when $\kappa \gg (Z_{LC}/\hbar)[S_{I_{qp}}(-\omega_0) - S_{I_{qp}}(\omega_0)] \sim (Z_{LC}/R_T)\omega_0$, yielding the same small parameter. For circuits having this separation of time scales, the approximation $\hat{I}_{qp}^H = 0$ in Eq. (7) then reproduces standard DCB expressions [37, 38] for phase fluctuations across the tunnel junction characterized by the impedance seen by the junction $\text{Re}Z_t(\omega) = Z_0\omega_0^4/|\Delta(\omega)|^2$. The resulting flux Φ is finally substituted in the current \hat{I}_{qp}^H in Eq. (6), enabling the calculation of spectral properties of the emitted light even for very strong DCB backaction. The net power carried out by the transmission line reads

$$P_{TL}(t) = \langle A_{\text{out}}^\dagger(t)A_{\text{out}}(t) \rangle - \langle A_{\text{in}}^\dagger(t)A_{\text{in}}(t) \rangle, \quad (8)$$

where $A_{\text{in/out}}(t) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\hbar\omega} a_{\text{in/out},\omega} e^{-i\omega t}$. Its calculation, taking the incoming field to be described by a displaced thermal state, agrees precisely with $\langle P_{LC}(t) \rangle$ in Eq. (1) in the high quality factor limit $\omega_0 \gg \kappa$. In particular, the products of the two terms in Eq. (6) mix

the field and the junction dynamics giving rise to the Bose factors in Eq. (1), which vanish at low temperature $\hbar\omega_0 \gg k_B T$, and to the Joule power dissipated in the junction.

Squeezed radiation - As strong DCB is responsible for non-linearities in transport, it is expected to also favor squeezing in the field emitted by the tunnel junction. The circuit of Fig. 1(b) is however not adapted to this study. An efficient squeezing requires to fully absorb the incoming mode, so that its trivial fluctuations do not pollute those of the outgoing field. This situation can be implemented by matching the impedance of the junction to its environment [26] but then the junction would shunt the fluctuations of the environment, thus reducing the effective non-linearities and hence the squeezing efficiency. We thus consider Fig. 1(c) where DCB and readout are spatially separated: on one side, the tunnel junction is coupled to a resistive circuit producing strong DCB, on the other side, a weakly coupled but impedance matched circuit is used to probe the radiation emitted by the junction. The LC resonator is necessary to achieve impedance matching such that photons incoming from the transmission line are all absorbed by the device and the output field measures only photons emitted by the tunnel junction.

We assume the following hierarchy of resistances $R_T \gg R \gg R_K \gg Z_0, Z_{LC}$. The high resistance R imposes strong fluctuations for Φ_B at the tunnel junction. An even larger R_T is necessary to avoid shunting those fluctuations. In contrast to that, the resonant circuit produces weak fluctuations giving a negligible contribution to DCB effects and the flux Φ can be expanded to first order yielding the inductive coupling $-\Phi I_{qp}$ in the Hamiltonian in which the tunnel coupling T is dressed by the flux Φ_B only. Hence, dynamics of the tunnel junction and DCB resistive circuit has been isolated, only weakly probed by the readout circuit.

The input-output theory is constructed similarly to Ref. [26]: time evolution is still described by Eqs. (4) while the current operator follows from linear response to the inductive coupling $\hat{I}_{qp}^H(\omega) = \hat{I}_{qp}(\omega) + i\omega\Phi(\omega)/Z_0(\omega)$. We have introduced the impedance of the junction whose real part R_0 is $2\hbar\omega/R_0(\omega) \equiv S_{I_{qp}}(-\omega) - S_{I_{qp}}(\omega)$, and coincides with the bare resistance $R_0(\omega) = R_T$ in the absence of DCB. We focus in what follows on the case of perfect impedance matching $R_0(\omega_0)Z_0 = Z_{LC}^2$. In frequency space, we find ($\delta\omega = \omega - \omega_0$)

$$a_{\text{out},\omega} = \frac{\delta\omega}{\delta\omega + i\kappa} a_{\text{in},\omega} - \sqrt{\frac{R_0(\omega_0)}{2\hbar\omega_0}} \frac{i\kappa}{\delta\omega + i\kappa} \hat{I}_{qp}(\omega), \quad (9)$$

which confirms that the input field is totally absorbed on resonance $\delta\omega = 0$. In this case, photon statistics in the output field are fully determined by current fluctuations in the tunnel junction. The formula (9) applies as long as the detection scheme induces small voltage

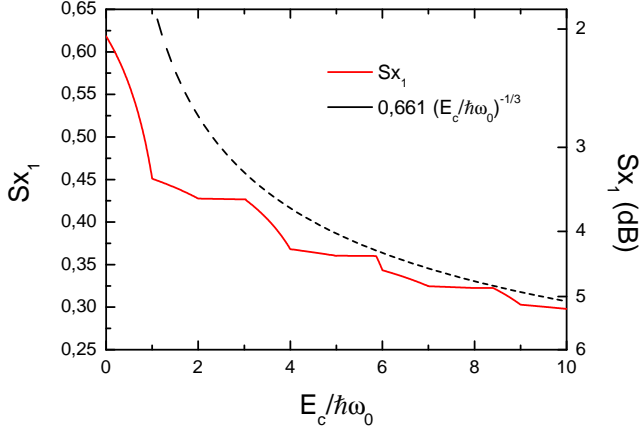


FIG. 2: From Eq. (10), squeezing of the output quadrature $i(a_{\text{out}} - a_{\text{out}}^\dagger)$, at the resonant frequency ω_0 , as function of the charging energy $E_c/(\hbar\omega_0)$ for very strong resistive DCB. The bias voltage at the junction is a sum of dc and ac components, $V(t) = V_{dc} + V_{ac} \cos(2\omega_0 t)$ chosen to optimize (minimize) S_{X_1} for each E_c . The asymptotic slow decay $0.661(E_c/\hbar\omega_0)^{-1/3}$ is reached at large $E_c/\hbar\omega_0$.

quantum fluctuations ($R_Q \gg Z_0, Z_{LC}$) but is generically valid for arbitrary DCB settings and tunnel couplings. For concreteness and in order to highlight squeezing effects, we consider the case of high impedance environment $R \gg R_Q$ at zero temperature. The function $P(E)$ [38], which gives the probability of the environment to absorb an energy E from a tunneling electron, has a simple form $P(E) = \delta(E - E_c)$, $E_c = e^2/(2C_B)$ is the charging energy. Equilibrium noise is then given by $S_{eq}(\omega) = -(2\hbar/R_T)(\omega + E_c/\hbar)\theta(-\omega - E_c/\hbar)$ with Heaviside function $\theta(x)$.

Squeezing is characterized by the power spectrum $S_{X_1}(\omega)$ of the quadrature $i(a_{\text{out},\omega_0+\omega} - a_{\text{out},\omega_0-\omega}^\dagger)$. Under parametric excitation, *i.e.* an ac voltage bias applied on the junction at frequency $2\omega_0$, we find at $\omega = 0$ [25]

$$S_{X_1} = \frac{\sum_n |c_n - c_{n+1}|^2 S_{eq,e} \left(\frac{eV_{dc}}{\hbar} + (2n+1)\omega_0 \right)}{\sum_n (|c_n|^2 - |c_{n+1}|^2) S_{eq,o} \left(\frac{eV_{dc}}{\hbar} + (2n+1)\omega_0 \right)}, \quad (10)$$

where we introduced the even and odd parts of the equilibrium noise S_{eq} . The coefficients $c_n = J_n(eV_{ac}/2\hbar\omega_0)$ are given in terms of Bessel functions J_n for a single-tone ac bias. The result for the squeezed quadrature is displayed Fig. 2, the $E_c/\hbar\omega_0 = 0$ limit agrees with the results obtained neglecting DCB effects [23, 25, 26], while it improves with the ratio $E_c/(\hbar\omega_0)$. Junctions with nanoscale cross-section [54] can implement charging energies as large as $E_c/\hbar = 4$ THz, while narrow-band high-impedance detection schemes having low characteristic impedance can be easily implemented in the few GHz range [55]. Ratios as large as 10^3 can be therefore expected, giving rise to promising squeezing levels of 12 dB. Perfect squeezing can even be reached by ex-

citing the junction with short and periodic Lorentzian pulses, also called a Leviton signal [56–59]. The frequency of arrival is $2\omega_0$ such that each pulse of duration τ reaches the junction when the quadrature to be squeezed is maximum. In this case, $c_{-1} = -e^{-2\omega_0\tau}$, $c_{n \geq 0} = e^{-2n\omega_0\tau}(1 - e^{-4\omega_0\tau})$, and $c_{n < -1} = 0$, leading to $S_{X_1} = \frac{1 - e^{-2\omega_0\tau}}{1 + e^{-2\omega_0\tau}}$ for $E_c > 3\hbar\omega_0 - eV_{dc}$. S_{X_1} therefore vanishes in the limit of instantaneous pulses $\tau\omega = 0$. Furthermore, the corresponding radiation is a pure squeezed state, meaning that the second quadrature has the minimal quantum uncertainty required by the Heisenberg principle $S_{X_1}S_{X_2} = 1$. In Ref. [25], similar results were obtained with a quantum dot.

In summary, we formulated a general input-output theory that captures at the same level strong dynamical Coulomb blockade physics and the quantum properties of the emitted light. We showed how strong blockade amplifies quadrature squeezing in the emitted field under parametric excitation. We gave specific results for the case of a tunnel junction but the generality of our approach makes it applicable to other conductors. The cases of quantum dots [2–4] and hybrid systems such as SIS junctions [60] seem particularly appealing for the purpose of squeezing efficiency. The extension of our approach to co-tunneling processes, where two electrons may cooperate to emit a photon [61, 62], is another promising direction.

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Supplementary information for the article: Input-output theory of Coulomb Blockade

These supplementary information provides the full calculations allowing us to derive the expressions present in the main article. Cited equations not preceded by $S-$ refer to the main text.

STANDARD $P(E)$ APPROACH

Electromagnetic power

The power emitted into the LC circuit is defined by the power operator

$$P_{LC} = dH_{LC}/dt = \frac{1}{2C} \left(\hat{I}_{qp} Q + Q \hat{I}_{qp} \right).$$

Here we compute the mean value of this operator up to lowest order in the coupling Hamiltonian H_T . Making use of the interaction picture of the power operator $P_{LC}^0(t)$ with respect to the uncoupled evolution $H_{qp} + H_{env}$, its time evolution up to first order in the tunnel coupling reads (here-after, the time-dependence of unlabeled operators are meant to be taken in the interaction picture):

$$P_{LC}^1(t) = P_{LC}^0(t) + \frac{i}{\hbar} \int_{-\infty}^0 [H_T(t+\tau), P_{LC}^0(t)] d\tau$$

Its quantum average over the initial states described in the article simplifies in:

$$\langle P_{LC}^1(t) \rangle = -\frac{1}{C} \text{Re} \int_{-\infty}^0 \frac{2e}{\hbar^2} \langle T(t+\tau) T^\dagger(t) Q(t) - T^\dagger(t+\tau) T(t) Q(t) \rangle - \langle \hat{I}_{qp}(t+\tau) \hat{I}_{qp}(t) \rangle d\tau, \quad (S1)$$

where we identified $\langle \hat{I}_{qp}(t+\tau) \hat{I}_{qp}(t) \rangle = \frac{e^2}{\hbar^2} \langle T(t+\tau) T^\dagger(t) + T^\dagger(t+\tau) T(t) \rangle$. Equation (S1) contains the real part of the already known quasiparticle current time-correlator, and two new correlation functions which we will now compute. Since the initial states are uncoupled, the correlation functions factorize in terms of quasiparticle and environment correlation functions. The quasiparticle correlation $\Theta(t+\tau)\Theta^\dagger(t)$, and $\Theta^\dagger(t+\tau)\Theta(t)$, where $\Theta = \sum_{l,r} \tau_{l,r} c_l^\dagger c_r$ is the quasiparticle tunneling operator, are already well known. In the case of a particle-hole symmetric systems (which is the case of metallic tunnel junctions probed in the relevant range of energies much smaller than the barrier height and Fermi energy), one has:

$$\theta(\tau) = \langle \Theta(t+\tau) \Theta^\dagger(t) \rangle = \langle \Theta^\dagger(t+\tau) \Theta(t) \rangle = \frac{\hbar G_T}{2\pi e^2} \left(i\pi \hbar \frac{d}{dt} \delta(\tau) - \frac{\pi^2}{\beta^2} \sinh^{-2} \left(\frac{\pi \tau}{\hbar \beta} \right) \right),$$

where G_T is the tunneling conductance, and β the inverse temperature. Therefore, we only need to compute the environment correlation functions: $\langle e^{\pm i e \Phi(t+\tau)/\hbar} e^{\mp i e \Phi(t)/\hbar} Q(t) \rangle$. For the sake of clarity, we first compute them for a dc bias, and then discuss how an ac bias modifies this first result.

dc bias

The magnetic flux operator reads: $\Phi(t) = V_{dc}t + \delta\Phi(t)$, with $\delta\Phi(t) = \frac{\hbar r}{e} (a(t) + a^\dagger(t))$, where $r = \sqrt{\frac{\pi Z_{LC}}{R_Q}}$ with $Z_{LC} = \sqrt{\frac{L}{C}}$ the mode impedance and $R_Q = h/e^2 \simeq 25.8 \text{ k}\Omega$ the resistance quantum, and where $a(t)$ and $a^\dagger(t)$ are correspondingly the mode annihilation and creation operators of the LC in the interaction picture. Therefore the operator $e^{\pm i e \delta\Phi(t)/\hbar}$ and the products $e^{\pm i e \delta\Phi(t+\tau)/\hbar} e^{\mp i e \delta\Phi(t)/\hbar}$ can be recast with displacement operators $D[\alpha] = e^{\alpha a^\dagger - \alpha^* a}$:

$$\begin{aligned} e^{\pm i e \delta\Phi(t)/\hbar} &= D[\pm i r e^{i\omega_0 t}], \\ e^{\pm i e \delta\Phi(t+\tau)/\hbar} e^{\mp i e \delta\Phi(t)/\hbar} &= e^{-ir^2 \sin(\omega_0 \tau)} D[\pm i r e^{i\omega_0 t} (e^{i\omega_0 \tau} - 1)]. \end{aligned}$$

From this, we can compute the correlation function $e^{J(\tau)} = \langle e^{\pm ie\delta\Phi(t+\tau)/\hbar} e^{\mp ie\delta\Phi(t)/\hbar} \rangle$ appearing in $P(E)$ theory, being its inverse Fourier transform:

$$\begin{aligned} \langle e^{\pm ie\delta\Phi(t+\tau)/\hbar} e^{\mp ie\delta\Phi(t)/\hbar} \rangle &= \frac{e^{-ir^2 \sin(\omega_0\tau)} e^{-\frac{1}{2}\beta\hbar\omega_0} e^{-\tau^2(1-\cos(\omega_0\tau))}}{Z_{LC}} \sum_n e^{-\beta n\hbar\omega_0} L_n^0(2r^2(1-\cos(\omega_0\tau))) \\ &= e^{J(\tau)} \end{aligned}$$

where $L_n^m(x)$ are generalized Laguerre polynomials of order n , from which we recover the well known expression for a single mode

$$J(\tau) = r^2 \left((\cos(\omega_0\tau) - 1) \coth\left(\frac{\beta\hbar\omega_0}{2}\right) - i \sin(\omega_0\tau) \right).$$

A similar calculation gives the result:

$$\langle e^{\pm ie\Phi(t+\tau)/\hbar} e^{\mp ie\Phi(t)/\hbar} Q(t) \rangle = \pm \frac{e}{2} e^{\pm ieV_{dc}\tau/\hbar} \left((1 - e^{i\omega_0\tau} + e^{\beta\hbar\omega_0} e^{-i\omega_0\tau}) e^{J(\tau)} - \frac{ie^{\beta\hbar\omega_0} (e^{J(\tau)})'}{\omega_0 r^2} \right). \quad (\text{S2})$$

Inserting Eq. (S2) back in Eq. (S1), together with the already known expression for quasiparticle current fluctuations $\langle \hat{I}_{qp}(t+\tau) \hat{I}_{qp}(t) \rangle = \frac{2e^2}{\hbar^2} \theta(\tau) e^{J(\tau)} \cos(eV_{dc}\tau/\hbar)$, the average electromagnetic power reads:

$$\begin{aligned} \langle P_{LC}^1(eV_{dc}) \rangle &= -\frac{2e^2}{\hbar^2 C} \text{Re} \int_{-\infty}^0 \cos(eV_{dc}\tau/\hbar) \theta(\tau) \left[(e^{\beta\hbar\omega_0} e^{-i\omega_0\tau} - e^{i\omega_0\tau}) e^{J(\tau)} - \frac{ie^{\beta\hbar\omega_0} (e^{J(\tau)})'}{\omega_0 r^2} \right] d\tau \\ &= \frac{1}{2C} \left((1 + n_B(\hbar\omega_0)) S_{I_{qp}}(V_{dc}, \omega_0) - n_B(\hbar\omega_0) S_{I_{qp}}(V_{dc}, -\omega_0) \right) \end{aligned} \quad (\text{S3})$$

which is the article Equation (1) specialized to the case of a dc bias. It can also take the following form:

$$\langle P_{LC}^1(eV_{dc}) \rangle = \int_{-\infty}^{+\infty} \frac{S_V(-\omega) S_{I_{qp}}(V_{dc}, \omega)}{\hbar\omega} \frac{d\omega}{2\pi},$$

where we introduced the spectral density of voltage fluctuations of the LC circuit:

$$S_V(\omega) = \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle \dot{\Phi}(t+\tau) \dot{\Phi}(t) \rangle - \langle \dot{\Phi}(t+\tau) \rangle \langle \dot{\Phi}(t) \rangle.$$

This expression is more symmetric in the sense that the power emitted (absorbed) from the tunnel junction via its current fluctuations is proportional to the spectral density of emission (absorption) current fluctuations of the junction multiplied by the spectral density of absorption (emission) voltage fluctuations of the load electromagnetic environment. Note however that this expression is not fully symmetric: while the voltage fluctuations appear via their closed cumulant, the current fluctuations appear via their raw moment.

ac bias

We now take the boundary condition described in the article,

$$\rho_{LC}(t \rightarrow -\infty) = D[\gamma] \frac{e^{-\beta H_{LC}}}{Z_{LC}} D^\dagger[\gamma],$$

describing a thermal field being displaced by a "classical" source. The displacement vector $\gamma = iV_{ac}\sqrt{C/(2\hbar\omega_0)}$ gives rise to a deterministic time-dependent ac voltage: $\langle V(t) \rangle = \text{Tr}(\rho_{LC}(t \rightarrow -\infty) \dot{\Phi}(t)) = V_{dc} + V_{ac} \cos(\omega_0 t)$, without perturbing the quantum and thermal voltage fluctuations. Again, we exploit the properties of displacement operators in order to get:

$$\langle \hat{I}_{qp}(t+\tau) \hat{I}_{qp}(t) \rangle = \frac{2e^2}{\hbar^2} \theta(\tau) e^{J(\tau)} \cos\left(\frac{eV_{dc}\tau}{\hbar} + \frac{eV_{ac}}{\hbar\omega_0} (\sin(\omega_0(t+\tau)) - \sin(\omega_0 t))\right)$$

and

$$\begin{aligned} & \langle e^{ie\Phi(t+\tau)/\hbar} e^{-ie\Phi(t)/\hbar} Q(t) - e^{-ie\Phi(t+\tau)/\hbar} e^{ie\Phi(t)/\hbar} Q(t) \rangle = \\ & e \cos \left(\frac{eV_{dc}\tau}{\hbar} + \frac{eV_{ac}}{\hbar\omega_0} (\sin(\omega_0(t+\tau)) - \sin(\omega_0 t)) \right) \left((1 - e^{i\omega_0\tau} + e^{\beta\hbar\omega_0} e^{-i\omega_0\tau}) e^{J(\tau)} - \frac{ie^{\beta\hbar\omega_0} (e^{J(\tau)})'}{\omega_0 r^2} \right) \\ & + 2i \sin \left(\frac{eV_{dc}\tau}{\hbar} + \frac{eV_{ac}}{\hbar\omega_0} (\sin(\omega_0(t+\tau)) - \sin(\omega_0 t)) \right) \theta(\tau) e^{J(\tau)} C V_{ac} \cos(\omega_0 t), \end{aligned}$$

From which we obtain the first equation of the article:

$$\begin{aligned} \langle P_{LC}^1(t) \rangle &= \frac{1}{2C} \left((1 + n_B(\hbar\omega_0)) S_I(\omega_0, t) - n_B(\hbar\omega_0) S_I(-\omega_0, t) \right) - \langle \hat{I}_{qp}^1(t) \rangle V_{ac} \cos(\omega_0 t) \\ &= \int_{-\infty}^{+\infty} \frac{S_V(-\omega) S_{I_{qp}}(\omega, t) d\omega}{\hbar\omega} \frac{1}{2\pi} - \langle \hat{I}_{qp}^1(t) \rangle V_{ac} \cos(\omega_0 t). \end{aligned}$$

Contrary to the stationary case, where power is exchanged only via the current and voltage fluctuations of the circuit, now the junction can also dissipate some energy initially contained in the LC circuit via the average time-dependent current response, as stressed by the last equality.

Joule power

We define now the power injected within the electrodes:

$$P_{qp} = \frac{i}{\hbar} [H_0, H_{qp}] = -\frac{i}{\hbar} \sum_{l,r} (\epsilon_l - \epsilon_r) \tau_{l,r} c_l^\dagger c_r e^{ie\Phi/\hbar} + (\epsilon_r - \epsilon_l) \tau_{l,r}^* c_r^\dagger c_l e^{-ie\Phi/\hbar},$$

and we expand its time evolution to first order in the tunnel coupling:

$$\begin{aligned} P_{qp}^1(t) &= P_{qp}^0(t) + \frac{2}{\hbar^2} \text{Re} \int_{-\infty}^0 \sum_{l,r} T^\dagger(t+\tau) (\epsilon_l - \epsilon_r) \tau_{l,r} c_l^\dagger c_r e^{i(\epsilon_l - \epsilon_r)t/\hbar} e^{ie\Phi(t)/\hbar} \\ &\quad - T(t+\tau) (\epsilon_l - \epsilon_r) \tau_{l,r}^* c_r^\dagger c_l e^{-i(\epsilon_l - \epsilon_r)t/\hbar} e^{-ie\Phi(t)/\hbar} d\tau. \end{aligned}$$

Again, the evaluation of this operator with the uncoupled boundary conditions factorizes in quasiparticle and environment correlation functions. The environment correlation functions are just the standard $\langle e^{\pm ie\Phi(t+\tau)/\hbar} e^{\mp ie\Phi(t)/\hbar} \rangle = e^{\pm i \left(\frac{eV_{dc}\tau}{\hbar} + \frac{eV_{ac}}{\hbar\omega_0} (\sin(\omega_0(t+\tau)) - \sin(\omega_0 t)) \right) e^{J(\tau)}}$. Then we only need to compute the new quasiparticle correlation functions:

$$\begin{aligned} \langle \Theta(t+\tau) \sum_{l,r} (\epsilon_l - \epsilon_r) \tau_{l,r}^* c_r^\dagger c_l e^{-i(\epsilon_l - \epsilon_r)t/\hbar} \rangle &= -i\hbar \dot{\theta}(\tau), \\ &= -\langle \Theta^\dagger(t+\tau) \sum_{l,r} (\epsilon_l - \epsilon_r) \tau_{l,r} c_l^\dagger c_r e^{i(\epsilon_l - \epsilon_r)t/\hbar} \rangle \end{aligned}$$

where the last equality holds for the particle-hole symmetric junctions.

Picking up the terms we obtain for a dc bias:

$$\langle P_{qp}^1(eV_{dc}) \rangle = \frac{2\pi}{\hbar} \left(\epsilon \theta(\epsilon) * P(\epsilon)_{(eV_{dc})} + \epsilon \theta(\epsilon) * P(\epsilon)_{(-eV_{dc})} \right),$$

while for an ac bias we get:

$$\langle P_{qp}^1(t) \rangle = \frac{2}{\hbar^2} \text{Re} \int_{-\infty}^0 d\tau 2 \cos \left(\frac{eV_{dc}\tau}{\hbar} + \frac{eV_{ac}}{\hbar\omega_0} (\sin(\omega_0(t+\tau)) - \sin(\omega_0 t)) \right) i\hbar \dot{\theta}(\tau) e^{J(\tau)}.$$

Power balances

dc bias

Since there is neither a dc voltage drop across the inductance, nor a dc displacement current, the average electrical power which is supplied by the voltage source is directly:

$$\begin{aligned} \langle \hat{I}_{qp}(eV_{dc}) \rangle V_{dc} &= \frac{2\pi}{\hbar} \int dE (eV_{dc} - E) \theta(E) P(eV_{dc} - E) + (-eV_{dc} - E) \theta(E) P(-eV_{dc} - E) \\ &+ \frac{2\pi}{\hbar} \int dE E \theta(E) P(eV_{dc} - E) + E \theta(E) P(-eV_{dc} - E), \end{aligned}$$

where we identify the power absorbed in the quasiparticles

$$P_{qp}(eV_{dc}) = \frac{2\pi}{\hbar} \int dE E \theta(E) P(eV_{dc} - E) + E \theta(E) P(-eV_{dc} - E).$$

The other term can be worked out to match the average power emitted into the environment:

$$\frac{2\pi}{\hbar} \int dE (eV_{dc} - E) \theta(E) P(eV_{dc} - E) + (-eV_{dc} - E) \theta(E) P(-eV_{dc} - E) = P_{LC}^1(eV_{dc})$$

With these identifications we obtain the stationary power balance:

$$P_{DC}(eV_{dc}) = P_{qp}(eV_{dc}) + P_{LC}(eV_{dc}).$$

ac bias

In the presence of the ac bias, the average power delivered by the dc source reads:

$$\overline{\langle \hat{I}_{qp}(t) \rangle V_{dc}} = \sum_k J_k^2 \left(\frac{eV_{ac}}{\hbar\omega_0} \right) I_{qp}(eV_{dc} + k\hbar\omega_0) V_{dc},$$

where we exploited the Jacobi-Angers expansion of exponentials having trigonometric arguments. On the other hand, the average dissipative ac response reads:

$$\begin{aligned} \overline{\langle \hat{I}_{qp}(t) \rangle V_{ac} \cos(\omega_0 t)} &= \sum_k J_k \left(\frac{eV_{ac}}{\hbar\omega_0} \right) \frac{V_{ac}}{2} \left(J_{k+1} \left(\frac{eV_{ac}}{\hbar\omega_0} \right) + J_{k-1} \left(\frac{eV_{ac}}{\hbar\omega_0} \right) \right) I_{qp}(eV_{dc} + k\hbar\omega_0) \\ &= \sum_k \frac{k\hbar\omega_0}{e} J_k^2 \left(\frac{eV_{ac}}{\hbar\omega_0} \right) I_{qp}(eV_{dc} + k\hbar\omega_0). \end{aligned}$$

Combining the two expression with the results for a dc bias, we get the power balance of the circuit:

$$\overline{\langle \hat{I}_{qp}(t) \rangle V_{dc}} = \overline{P_{qp}(t)} + \overline{P_{LC}(t)}.$$

POWER RADIATED IN THE TRANSMISSION LINE

Each node j of the circuit is represented by conjugated quantum variables: the local charge Q_j and flux Φ_j with the commutation relation $[\Phi_j, Q_{j'}] = i\hbar\delta_{j,j'}$. Using this representation, time-evolution is described by the Heisenberg equations (at each node) $\partial_t Q_j = -\frac{\partial H}{\partial \Phi_j}$ and $\partial_t \Phi_j = \frac{\partial H}{\partial Q_j}$, leading to Eqs. (4) in the main text. Note that no charge can accumulate at node A in Fig.1(b-c) and we simply have $\frac{\partial H}{\partial \Phi_A} = 0$. The input-output formulation is obtained in frequency space, given by Eq. (6) and

$$\Phi(\omega) = -\omega_0^2 \sqrt{\frac{\hbar Z_0}{2\omega}} \frac{2a_{in,\omega} + \left(\frac{i}{\omega} + \frac{1}{\kappa} \right) \sqrt{\frac{2\omega Z_0}{\hbar}} \hat{I}_{qp}^H(\omega)}{\Delta(\omega)}, \quad (S4)$$

simplified as Eq. (7) in the main text. Neglecting \hat{I}_{qp}^H and using a thermal distribution for the input field, $\langle a_{\text{in},\omega}^\dagger a_{\text{in},\omega'} \rangle = 2\pi n_B(\hbar\omega)\delta(\omega - \omega')$, we recover standard DCB expressions for phase fluctuations across the tunnel junction

$$\langle e^{ie\Phi(t)/\hbar} e^{-ie\Phi(0)/\hbar} \rangle \equiv e^{J(t)} \quad J(t) = 2 \int_0^{+\infty} \frac{d\omega}{\omega} \frac{\text{Re}Z_t(\omega)}{R_Q} \left[\coth\left(\frac{\beta\omega}{2}\right) (\cos\omega t - 1) - i \sin\omega t \right], \quad (\text{S5})$$

where $\beta = 1/(k_B T)$ is the inverse temperature of the input field and $\text{Re}Z_t(\omega) = Z_0\omega_0^4/|\Delta(\omega)|^2$ is the real part of the impedance seen by the junction.

Let us consider first the absence of an ac bias voltage. The power injected in the output field is expressed as

$$\langle A_{\text{out}}^\dagger(t) A_{\text{out}}(t) \rangle = \int_0^{+\infty} d\omega \hbar\omega f_{\text{out}}(\omega) \quad (\text{S6})$$

where we introduced the photon-flux density from $\langle a_{\text{out},\omega}^\dagger a_{\text{out},\omega'} \rangle = 2\pi f_{\text{out}}(\omega)\delta(\omega - \omega')$. Defining the scattering phase $e^{i\theta_\omega} = \frac{\Delta^*(\omega)}{\Delta(\omega)}$ and the normalization factor $\mathcal{N}_\omega = \frac{\omega_0^2}{i\Delta(\omega)} \sqrt{\frac{2Z_0}{\hbar\omega}}$, the decomposition Eq. (6) (main text) of the input field produces four terms in the calculation of $f_{\text{out}}(\omega)$

$$\langle a_{\text{out},\omega}^\dagger a_{\text{out},\omega'} \rangle = e^{i(\theta_{\omega'} - \theta_\omega)} \langle a_{\text{in},\omega}^\dagger a_{\text{in},\omega'} \rangle + \mathcal{N}_{\omega'} e^{-i\theta_\omega} \langle a_{\text{in},\omega}^\dagger \hat{I}_{qp}^H(\omega') \rangle + \mathcal{N}_\omega^* e^{i\theta_{\omega'}} \langle \hat{I}_{qp}^H(-\omega) a_{\text{in}}(\omega') \rangle + \mathcal{N}_\omega \mathcal{N}_{\omega'}^* \langle \hat{I}_{qp}^H(-\omega) \hat{I}_{qp}^H(\omega') \rangle. \quad (\text{S7})$$

1. The first term is readily calculated

$$e^{i(\theta_{\omega'} - \theta_\omega)} \langle a_{\text{in},\omega}^\dagger a_{\text{in},\omega'} \rangle = 2\pi\delta(\omega - \omega') n_B(\omega) = \langle a_{\text{in},\omega}^\dagger a_{\text{in},\omega} \rangle. \quad (\text{S8})$$

It is equal to the photon-flux of the incoming field $a_{\text{in},\omega}$. This term is subtracted in the net output power $P_{LT} = \langle A_{\text{out}}^\dagger A_{\text{out}} \rangle - \langle A_{\text{in}}^\dagger A_{\text{in}} \rangle$ (Eq. (8) in the main text).

2. The last term is written in terms of the power spectral density of quasiparticle current fluctuations $S_{I_{qp}}(\omega)$. It takes the form

$$\mathcal{N}_\omega \mathcal{N}_{\omega'}^* \langle \hat{I}_{qp}^H(-\omega) \hat{I}_{qp}^H(\omega') \rangle = |\mathcal{N}_\omega|^2 S_{I_{qp}}(\omega) 2\pi\delta(\omega - \omega'), \quad (\text{S9})$$

where only the first term is kept in the expansion of \hat{I}_{qp}^H , see Eq. (5) of the main text. If we use this result in the expression of the radiated power Eq. (S6), we arrive at the contribution $S_{I_{qp}}(\omega_0)/(2C)$ under the assumption of a sharp resonance $\kappa \ll \omega_0$ and the integral

$$\int_0^{+\infty} d\omega \hbar\omega |\mathcal{N}_\omega|^2 \simeq \frac{Z_0\omega_0^2}{2\kappa} = \frac{1}{2C}, \quad (\text{S10})$$

in agreement with the prefactor in Eq. (1) of the main text. $S_{I_{qp}}(\omega_0)$ is interpreted as the emission noise corresponding to the power emitted by current fluctuations in the tunnel junction, even in the absence of the input field. It takes into account the influence of DCB on transport and the noise can be written as a convolution $S_{I_{qp}}(\omega) = \int d\varepsilon P(\varepsilon) S_{I_{qp}}^0(\omega - \varepsilon/\hbar)$ between the energy distribution function $P(E) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dt e^{J(t) + iEt/\hbar}$ and the noise in absence of DCB effect

$$S_{I_{qp}}^0(\omega) = \frac{1}{R_T} \sum_{\pm} \frac{\hbar\omega \pm eV_{dc}}{e^{\beta(\hbar\omega \pm eV_{dc})} - 1}. \quad (\text{S11})$$

3. The second and third terms in Eq. (S7) are complex conjugate to each other. In contrast to the last term in Eq. (S7), it is now the second term in the expansion of Eq. (5) (main text) which contributes to the calculation. The first term in Eq. (5) creates electron-hole excitations across the junction and has a vanishing expectation.

In the calculation, we make use of the following identity

$$\langle a_{\text{in}}^\dagger(t) [H_T(t_1), \hat{I}_{qp}(t_2)] \rangle = \left(\langle a_{\text{in}}^\dagger(t) \Phi(t_2) \rangle - \langle a_{\text{in}}^\dagger(t) \Phi(t_1) \rangle \right) \langle [\hat{I}_{qp}(t_1), \hat{I}_{qp}(t_2)] \rangle, \quad (\text{S12})$$

which is valid because the field a_{in} , and therefore $\Phi(\omega) = -i\hbar\mathcal{N}_\omega a_{\text{in}}(\omega)$, have Gaussian distributions. After a tedious but straightforward calculation, we find the additional contribution $n_B(\omega)(S_{I_{qp}}(\omega) - S_{I_{qp}}(-\omega))|\mathcal{N}_\omega|^2$ to $f_{\text{out}}(\omega)$. This term vanishes at zero temperature. We then integrate over frequencies using the integral Eq. (S10) and obtain the correction to the power in the output field

$$\frac{n_B(\omega) [S_{I_{qp}}(\omega) - S_{I_{qp}}(-\omega)]}{2C}. \quad (\text{S13})$$

To summarize, adding all contributions from Eq. (S7), the calculation of the net output power P_{LT} in the input-output formalism coincides with the power $\langle P_{LC}(t) \rangle$ received by the LC resonator in the standard DCB approach.

The presence of an ac voltage can be included rigorously in the quantum formalism thanks to the displacement operator D acting on both frequencies ω_0 and $-\omega_0$. Quantum averages are then taken with respect to the displaced density operator $\rho = D e^{-\beta H} D^\dagger / Z$ and the action on the input field is given by

$$D^\dagger a_{\text{in},\omega} D = a_{\text{in},\omega} + \frac{V_{ac}}{2Z_{LC}} \sqrt{\frac{Z_0}{2\hbar\omega_0}} [2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)]. \quad (\text{S14})$$

Using Eq. (S4) above, we obtain the shift in the flux

$$D^\dagger \Phi(\omega) D = \Phi(\omega) + \frac{iV_{ac}}{2\omega_0} [2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)], \quad (\text{S15})$$

leading to $D^\dagger \frac{\partial \Phi(t)}{\partial t} D = \frac{\partial \Phi(t)}{\partial t} + V_{ac} \cos(\omega_0 t)$. Instead of dressing the density operator with D , it is possible to work with the undisplaced density operator $e^{-\beta H} / Z$ while all operators of the theory are dressed by D and D^\dagger . The input-output relation (6) of the main text is then transformed to

$$a_{\text{out},\omega} = \frac{\Delta^*(\omega)}{\Delta(\omega)} a_{\text{in},\omega} - \frac{V_{ac}}{2Z_{LC}} \sqrt{\frac{Z_0}{2\hbar\omega_0}} [2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)] - i\omega_0^2 \sqrt{\frac{2Z_0}{\hbar\omega}} \frac{\hat{I}_{qp}^H(\omega)}{\Delta(\omega)}, \quad (\text{S16})$$

where the flux in the current $\hat{I}_{qp}^H(t)$ contains the classical evolution $\frac{V_{ac}}{\omega_0} \sin(\omega_0 t)$. Inserting this result into the expression of the net injected power $P_{TL}(t)$ retrieves Eq. (1) of the main text.

WEAK PROBE OF DCB

We shortly review the case of a tunnel junction weakly probed by the readout transmission line. Weak probe here means that the readout backaction on the junction is negligible. Since photon backaction (DCB) involves a weighted sum over all frequencies, this does not prevent the transmission line to be matched in impedance with the tunnel junction close to a resonant frequency.

The input-output theory is constructed starting with the Heisenberg Eqs. (4) in the main text. The Heisenberg current operator $\hat{I}_{qp}^H(t) = e^{iHt/\hbar} \hat{I}_{qp} e^{-iHt/\hbar}$ is evolved with the full Hamiltonian $H = H_0 + H_{\text{line}}$. At weak coupling, the flux Φ of the capacitance, see Fig. 1(c) in the main text, can be expanded to first order, thereby reducing the device-transmission line interaction to a linear inductive coupling, $H \simeq H^u - \Phi \hat{I}_{qp}$. The uncoupled evolution is set by the Hamiltonian $H^u = H_{qp} + H_T^u + H_{LC}$ where the tunnel coupling and the current operator change to

$$H_T^u = T^u + T^{u\dagger}, \quad T^u = \sum_{l,r} \tau_{l,r} c_l^\dagger c_r e^{-ie\Phi_B/\hbar}, \quad \hat{I}_{qp} = \frac{ie}{\hbar} (T^{u\dagger} - T^u), \quad (\text{S17})$$

and involve only the flux Φ_B . Further assuming weak inductive coupling, the time evolution of the Heisenberg current can be obtained to first order

$$\hat{I}_{qp}^H(t) = \hat{I}_{qp}(t) - \frac{i}{\hbar} \int_{-\infty}^t dt' \Phi(t') [\hat{I}_{qp}(t'), \hat{I}_{qp}(t)], \quad (\text{S18})$$

with the interaction picture current operator $\hat{I}_{qp}(t) = e^{iH^u t/\hbar} \hat{I}_{qp} e^{-iH^u t/\hbar}$. Weak coupling also justifies a quantum average of the commutator in Eq. (S18). This average includes the electronic variables in the lead, the photons associated to the flux Φ_B but not Φ . The result takes the form in frequency space

$$\hat{I}_{qp}^H(\omega) = \hat{I}_{qp}(\omega) + \frac{i\omega}{Z(\omega)} \Phi(\omega), \quad (\text{S19})$$

corresponding to a resistive dissipation in the tunnel junction with the resistance $R_0(\omega) = \text{Re}Z(\omega) = 2\hbar\omega [S_{I_{qp}}(-\omega) - S_{I_{qp}}(\omega)]^{-1}$. This last Eq. (S19) is the missing piece needed to complete the input-output calculation using the Eqs. (4) in the main text. The output field can now be written in terms of the input field and of the current fluctuations in the tunnel junction decoupled from the transmission line, namely

$$a_{\text{out},\omega} = \frac{\omega - \omega_0 + \frac{i}{2R(\omega)L}}{\omega - \omega_0 + \frac{i}{2R(\omega)L}} \frac{[Z_{LC}^2 - Z_0 R(\omega)]}{[Z_{LC}^2 + Z_0 R(\omega)]} a_{\text{in},\omega} - i\sqrt{\frac{Z_0\omega_0}{2\hbar}} \frac{\hat{I}_{qp}(\omega)}{\omega - \omega_0 + \frac{i}{2R(\omega)L} [Z_{LC}^2 + Z_0 R(\omega)]}. \quad (\text{S20})$$

The input field is fully absorbed at resonance $\omega = \omega_0$ and for impedance matching $Z_{LC}^2 = Z_0 R(\omega_0)$.

SQUEEZING WITH A SINGLE TONE

We study quadrature squeezing in the output field in the presence of a sinusoidal ac-bias $V_{ac}(t) = V_{ac} \cos(2\omega_0 t)$, oscillating at twice the resonant frequency of the LC circuit. The standard Fourier decomposition

$$e^{i \frac{eV_{ac}}{2\hbar\omega_0} \sin(2\omega_0 t)} = \sum_{m \in \mathbb{Z}} J_m \left(\frac{eV_{ac}}{2\hbar\omega_0} \right) e^{2im\omega_0 t}, \quad (\text{S21})$$

introducing the Bessel functions J_n , is used to derive the photo-assisted noise, or current-current correlator,

$$\begin{aligned} \langle \hat{I}_{qp}(\omega_1) \hat{I}_{qp}(\omega_2) \rangle = \sum_{n, m \in \mathbb{Z}} & \left\{ J_m \left(\frac{eV_{ac}}{2\hbar\omega_0} \right) J_{m+n} \left(\frac{eV_{ac}}{2\hbar\omega_0} \right) S_{eq}(-\omega_1 - eV_{dc}/\hbar - 2m\omega_0) \right. \\ & \left. + J_m \left(\frac{eV_{ac}}{2\hbar\omega_0} \right) J_{m-n} \left(\frac{eV_{ac}}{2\hbar\omega_0} \right) S_{eq}(-\omega_1 + eV_{dc}/\hbar + 2m\omega_0) \right\} \pi \delta(\omega_1 + \omega_2 - 2n\omega_0). \end{aligned} \quad (\text{S22})$$

The equilibrium noise includes DCB with, at zero temperature,

$$S_{eq}(\omega) = -\frac{2\hbar}{R_T} (E_c/\hbar + \omega) \theta(-E_c/\hbar - \omega). \quad (\text{S23})$$

We now introduce the quadrature $X_1(\omega) = i(a_{\text{out}, \omega_0 + \omega} - a_{\text{out}, \omega_0 - \omega}^\dagger)$ and its power spectrum $\langle X_1(\omega) X_1(\omega') \rangle = S_{X_1}(\omega) 2\pi \delta(\omega + \omega')$. On resonance, Eq. (9) in the main text gives $a_{\text{out}, \omega_0} = -\sqrt{R_0(\omega_0)/(2\hbar\omega_0)} \hat{I}_{qp}(\omega_0)$. We finally obtain Eq. (10) in the main text, or

$$S_{X_1}(0) = \frac{\sum_n |c_{n+1} - c_n|^2 S_{eq,e} \left(\frac{eV_{dc}}{\hbar} + (2n+1)\omega_0 \right)}{\sum_n (|c_{n+1}|^2 - |c_n|^2) S_{eq,o} \left(\frac{eV_{dc}}{\hbar} + (2n+1)\omega_0 \right)}, \quad (\text{S24})$$

with $c_n = J_n(eV_{ac}/2\hbar\omega_0)$, and $S_{eq,e/o}$ are respectively the even and odd parts of S_{eq} .

The function $S_{eq}(\omega)$ has a cusp at $\omega = E_c/\hbar$. In trying to minimize $S_{X_1}(0)$ numerically, one observes that the dc voltage V_{dc} finds this cusp for a specific value of n . This value itself depends on E_c/\hbar , hence the optimal voltage defines a piecewise function

$$eV_{dc}^{opt} = E_c + \begin{cases} \hbar\omega_0 & 0 < E_c/\hbar\omega_0 < 3.036 \\ -\hbar\omega_0 & 3.036 < E_c/\hbar\omega_0 < 5.868 \\ -3\hbar\omega_0 & 5.868 < E_c/\hbar\omega_0 < 8.408 \\ \dots & \end{cases} \quad (\text{S25})$$

Once this optimal value of V_{dc} is set, we still have to optimize squeezing with respect to V_{ac} . The result is shown Fig. 3 together with the corresponding $S_{X_1}(0)$.

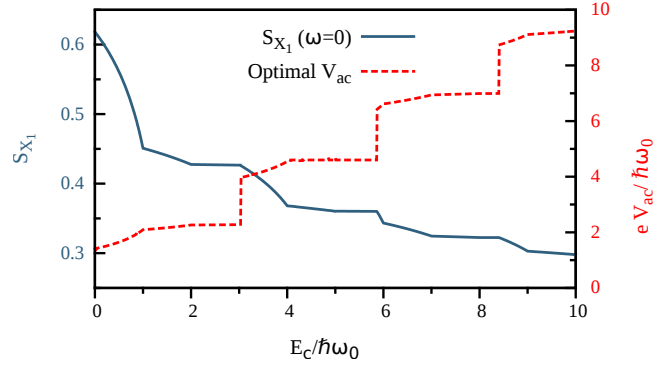


FIG. 3: (right) Optimal value of the ac bias voltage V_{ac} , in units of $\hbar\omega_0/e$, as function of the ratio $E_c/(\hbar\omega_0)$. (left) Corresponding minimized quadrature squeezing $S_{X_1}(0)$.